

# Grušin operators, Riesz transforms and nilpotent Lie groups

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## Abstract

We establish that the Riesz transforms of all orders corresponding to the Grušin operator  $H_N = -\nabla_x^2 - |x|^{2N} \nabla_y^2$ , and the first-order operators  $(\nabla_x, x^\nu \nabla_y)$  where  $x \in \mathbf{R}^n$ ,  $y \in \mathbf{R}^m$ ,  $N \in \mathbf{N}_+$ , and  $\nu \in \{1, \dots, n\}^N$ , are bounded on  $L_p(\mathbf{R}^{n+m})$  for all  $p \in \langle 1, \infty \rangle$  and are also weak-type  $(1, 1)$ . Moreover, the transforms of order less than or equal to  $N+1$  corresponding to  $H_N$  and the operators  $(\nabla_x, |x|^N \nabla_y)$  are bounded on  $L_p(\mathbf{R}^{n+m})$  for all  $p \in \langle 1, \infty \rangle$ . But all transforms of order  $N+2$  are bounded if and only if  $p \in \langle 1, n \rangle$ . The proofs are based on the observation that the  $(\nabla_x, x^\nu \nabla_y)$  generate a finite-dimensional nilpotent Lie algebra, the corresponding connected, simply connected, nilpotent Lie group is isometrically represented on the spaces  $L_p(\mathbf{R}^{n+m})$  and  $H_N$  is the corresponding sublaplacian

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# 1 Introduction

The primary aim of this note is to prove the boundedness of the Riesz transforms corresponding to the Grušin operators

$$H_N = -\nabla_x^2 - |x|^{2N} \nabla_y^2, \quad (1)$$

and the family of first-order operators  $(\nabla_x, x^\nu \nabla_y)$  where  $x \in \mathbf{R}^n$ ,  $y \in \mathbf{R}^m$ ,  $N \in \mathbf{N}_+$  and the multi-index  $\nu$  takes all values in the set  $\{1, \dots, n\}^N$ . We establish that the transforms of arbitrary order are bounded on the spaces  $L_p(\mathbf{R}^{n+m})$  for all  $p \in \langle 1, \infty \rangle$  and are also weak-type  $(1, 1)$ . The proof of boundedness is based on the observation that there is an underlying symmetry which arises because the operators  $(\nabla_x, x^\nu \nabla_y)$  generate a finite-dimensional nilpotent Lie algebra. This allows one to identify the transforms with the Riesz transforms associated with an isometric representation of a nilpotent Lie group and an affiliated subelliptic operator. It is known, however, that the Riesz transforms corresponding to very general subcoercive operators in the left, or right, regular representation of a nilpotent Lie group are bounded [BER94] [ERS97]. We will show that one can adapt the Coifman-Weis transference techniques [CW71] to deduce from this result that the transforms in a every isometric representation of the group are also bounded. As an immediate corollary one deduces that the Riesz transforms connected to the Grušin operators are bounded. This result was established recently by Jotsaroop, Sanjay and Thangavelu [JST13] for the special case  $m = 1 = N$  using quite different arguments (see also [ABCS13]).

The symmetry techniques that we exploit are sensitive to the choice of the first-order operators  $(\nabla_x, x^\nu \nabla_y)$  and are also dependent on the assumption that  $N$  is an integer. In order to gain some insight into the difficulties that arise in the non-integer situation we next examine an alternative choice of transforms. We consider the transforms corresponding to  $H_N$  and the first-order operators  $(\nabla_x, |x|^N \nabla_y)$ . These operators no longer generate a Lie algebra. Nevertheless, if  $n \geq 2$  one can deduce from the group-theoretic results that the corresponding Riesz transforms of all orders less than or equal to  $N + 1$  are bounded on  $L_p(\mathbf{R}^{n+m})$  with  $p \in \langle 1, \infty \rangle$  and also deduce that these transforms are weak-type  $(1, 1)$ . In addition one can demonstrate that some of the transforms of order  $N + 2$  are unbounded on the  $L_p$ -spaces with  $p \geq n$  but all the transforms of order  $N + 2$  are bounded on the  $L_p$ -spaces with  $p \in \langle 1, n \rangle$ .

In Section 2 we describe briefly the necessary background on Riesz transforms on nilpotent Lie groups and establish a very general boundedness result for isometric group representations. Theorem 2.3 and its corollaries provide the basis for the discussion of the Grušin transforms in Section 3 although the details of their proofs are largely superfluous for comprehension of the Grušin analysis.

## 2 Riesz transforms and nilpotent groups

Let  $G$  be a  $d$ -dimensional connected nilpotent Lie group with Haar measure  $dg$  and Lie algebra  $\mathfrak{g}$ . Further let  $r$  denote the rank of  $\mathfrak{g}$ . Fix an algebraic basis  $a_1, \dots, a_{d'}$  and a corresponding vector space basis  $a_1, \dots, a_{d'}, a_{d'+1}, \dots, a_d$  of  $\mathfrak{g}$ . The group acts isometrically by left translations  $L$  on the spaces  $L_p(G) = L_p(G; dg)$  for all  $p \in [1, \infty]$ . Next let  $A_i = dL(a_i)$  denote the corresponding generators of left translations, i.e.  $A_i$  is the generator

of the one-parameter group  $u_i \in \mathbf{R} \mapsto L(\exp(u_i a_i))$ , where  $u \in \mathbf{R}^d \mapsto \exp(u.a) \in G$  denotes the usual exponential map. We use multi-index notation for products of the generators corresponding to the algebraic basis. For example, if  $\alpha = (i_1, \dots, i_n)$  with  $i_j \in \{1, \dots, d'\}$  then  $A^\alpha = A_{i_1} \dots A_{i_n}$  and  $|\alpha| = n$ . We also set  $L'_{p;n}(G) = \bigcap_{\{\alpha: |\alpha| \leq n\}} D(A^\alpha)$  with the usual graph norm.

Since we wish to apply various results from [ERS97] it is also convenient to introduce the nilpotent Lie group  $\tilde{G}$  with  $d'$  generators which is free of step  $r$ . First define  $\tilde{\mathfrak{g}}$  as the quotient of the free Lie algebra with  $d'$  generators  $\tilde{a}_1, \dots, \tilde{a}_{d'}$  by the ideal generated by the commutators of order at least  $r+1$ . Then  $\tilde{G}$  is the connected simply connected Lie group with Lie algebra  $\tilde{\mathfrak{g}}$ . It is non-compact, stratified and acts isometrically by left translations  $\tilde{L}$  on the spaces  $L_p(\tilde{G}) = L_p(\tilde{G}; d\tilde{g})$  where  $d\tilde{g}$  denotes the Haar measure of  $\tilde{G}$ .

Next let  $X = (X, \mu)$  be a  $\sigma$ -finite measure space and  $U$  an isometric representation of  $G$  on the spaces  $L_p(X) = L_p(X; \mu)$ . Further let  $D_k = dU(a_k)$  denote the corresponding group generators,  $D^\beta$  their products and set  $L'_{p;n}(X) = \bigcap_{\{\beta: |\beta| \leq n\}} D(D^\beta)$ . Then for each  $k \in L_1(G)$  we define bounded operators  $K_L$  and  $K_U$  acting on  $L_p(G)$  and  $L_p(X)$ , respectively by

$$K_L = \int_G dg k(g) L(g) \quad \text{and} \quad K_U = \int_G dg k(g) U(g).$$

Note that the  $K_L$  act by convolution,  $K_L \varphi = k * \varphi$ , where  $(k * \varphi)(g) = \int_G dh k(g) \varphi(h^{-1}g)$ .

One can transfer estimates on  $K_L$  to  $K_U$  by the Coifman–Weiss transference theorem.

**Proposition 2.1** *If  $k \in L_1(G)$  then  $\|K_U\|_{L_p(X) \rightarrow L_p(X)} \leq \|K_L\|_{L_p(G) \rightarrow L_p(G)}$  for all  $p \in [1, \infty]$ .*

If  $k$  has compact support then the result is given by Theorem 2.4 of [CW71]. The general result follows by approximating  $k$  with compactly supported functions and using the isometry of the group actions  $U$  and  $L$ .

Next observe that

$$\begin{aligned} U(\exp(u.a))K_U &= \int_G dg k(g) U(\exp(u.a)g) \\ &= \int_G dg k(\exp(-u.a)g) U(g) = \int_G dg (L(\exp(u.a))k)(g) U(g) \end{aligned}$$

by the group property and invariance of Haar measure. Therefore if  $k \in L'_{1;n}(G)$  then  $K_U L_p(X) \subseteq L'_{p;n}(X)$  and

$$D^\alpha K_U = \int_G dg (A^\alpha k)(g) U(g) \tag{2}$$

for all  $\alpha$  with  $|\alpha| \leq n$ .

Let  $p \in \langle 1, \infty \rangle$  and let  $m$  be an even positive integer. Following [ERS97] we introduce homogeneous  $m$ -th order operators

$$H = \sum_{|\alpha|=m} c_\alpha A^\alpha \quad \text{and} \quad \tilde{H} = \sum_{|\alpha|=m} c_\alpha \tilde{A}^\alpha,$$

with  $c_\alpha \in \mathbf{C}$ , on  $L'_{p;m}(G)$  and  $L'_{p;m}(\tilde{G})$ , respectively. These operators are closable and for simplicity we assume that  $H$  and  $\tilde{H}$  denote the closures. Further we assume  $H$  is

subcoercive of step  $r$  in the sense of [ER94], i.e.  $\tilde{H}$  satisfies the Gårding inequality

$$\operatorname{Re}(\tilde{\varphi}, \tilde{H}\tilde{\varphi}) \geq \mu \sum_{|\alpha|=m/2} \|\tilde{A}^\alpha \tilde{\varphi}\|_2^2$$

for some  $\mu > 0$  uniformly for all  $\tilde{\varphi} \in C_c^\infty(\tilde{G})$ .

**Remark 2.2** The subcoercivity requirement for second-order operators with  $r \geq 2$  is equivalent to the assumption that  $\Re C = 2^{-1}(C + C^*) \geq \mu I > 0$ , where  $C$  is the  $d' \times d'$ -matrix given by  $C_{ij} = -c_{(i,j)}$  (see [ER95], Proposition 3.7). In particular the sublaplacian  $-\sum_{j=1}^{d'} A_j^2$  is subcoercive.

It follows that each subcoercive operator  $H$  generates a holomorphic semigroup  $S_t$  with a  $C^\infty$ -kernel  $k_t$  on  $L_p(G)$ . Specifically

$$S_t = \int_G dg k_t(g) L(g)$$

for all  $t > 0$  or, in the previous notation,  $S_t = k_{t,L}$ . (We refer to [ER94] and [ER95] for details on the semigroup and its kernel.) Next for each  $t > 0$  we define the operator

$$S_t^U = \int_G dg k_t(g) U(g) \quad (= k_{t,U})$$

on  $L_p(X)$ . Since  $k_t$  is a convolution semigroup and  $U$  a group representation it follows that  $S^U$  is a semigroup. Moreover,  $\|S_t^U\|_{L_p(X) \rightarrow L_p(X)} \leq \|S_t\|_{L_p(G) \rightarrow L_p(G)}$  by Proposition 2.1. Now let

$$H_D = \sum_{|\alpha|=m} c_\alpha D^\alpha$$

on  $L'_{p;m}(X)$ . It follows from (2) that  $S_t^U L_p(X) \subseteq L'_{p;m}(X)$  for all  $t > 0$  and

$$H_D S_t^U = \int_G dg (H k_t)(g) U(g) = -\frac{d}{dt} S_t^U$$

for all  $t > 0$ . Therefore the closure of  $H_D$ , which for simplicity we also denote by  $H_D$ , is the generator of  $S^U$ . Moreover,  $\|H_D S_t^U\|_{L_p(X) \rightarrow L_p(X)} \leq \|H S_t\|_{L_p(G) \rightarrow L_p(G)}$  by another application of Proposition 2.1. Hence  $S^U$  inherits the holomorphy properties of  $S$ .

Now we are prepared to prove boundedness of the Riesz transforms corresponding to the derivatives  $D$  and the operator  $H_D$ . These transforms are formally given by  $R_\alpha^U = D^\alpha H_D^{-|\alpha|/m}$  on  $L_p(X)$  but it is not clear that the operators are even densely-defined. In fact the operators are given by integral kernels which are logarithmically divergent both at the identity and at infinity. These difficulties can, however, be overcome by the techniques used in earlier papers [BER94] [ERS97] to handle the Riesz transforms  $R_\alpha$  and  $\tilde{R}_\alpha$  associated with  $H$  and  $\tilde{H}$  on  $L_p(G)$  and  $L_p(\tilde{G})$ , respectively. Theorem 4.4 of [ERS97] establishes that the  $R_\alpha$  and  $\tilde{R}_\alpha$  extend to bounded operators but the same is true for the  $R_\alpha^U$ .

**Theorem 2.3** *The Riesz transforms  $R_\alpha^U$  extend to bounded operators on  $L_p(X)$  for each  $p \in \langle 1, \infty \rangle$ . Moreover, there exist  $a_p, b_p > 0$  such that*

$$\|R_\alpha^U\|_{L_p(X) \rightarrow L_p(X)} \leq a_p b_p^k \|R_\alpha\|_{L_p(G) \rightarrow L_p(G)} \quad (3)$$

where  $k$  is the integer part of  $|\alpha|/m$ .

**Proof** The theorem is a direct analogue of Theorem 4.4 of [ERS97] and it follows by an adaptation of the arguments of [BER94] [ERS97] supplemented by the transference statement of Proposition 2.1. But first note that if  $G$  is compact then we define the transforms  $R_\alpha$  to be zero on the constant functions and analyze the transforms on  $L_p(G)$  modulo the constant functions.

The proof of the theorem is based on estimation of the regularized transforms

$$R_{\alpha;\nu,\varepsilon} = A^\alpha(\nu I + H)^{-|\alpha|/m}(I + \varepsilon H)^{-n}$$

on  $L_p(G)$  with  $\varepsilon, \nu > 0$  and  $n$  a large positive integer together with the corresponding regularizations  $R_{\alpha;\nu,\varepsilon}^U$  of  $R_\alpha^U$  on  $L_p(X)$ .

The first point is that  $R_{\alpha;\nu,\varepsilon}$  and  $R_{\alpha;\nu,\varepsilon}^U$  are given by a kernel  $k_{\alpha;\nu,\varepsilon} \in L_1(G)$ . The introduction of the  $\nu$ -term ensures integrability at infinity and the factor  $(I + \varepsilon H)^{-n}$  ensures integrability at the identity. Consequently

$$\|R_{\alpha;\nu,\varepsilon}^U\|_{L_p(X) \rightarrow L_p(X)} \leq \|R_{\alpha;\nu,\varepsilon}\|_{L_p(G) \rightarrow L_p(G)} \quad (4)$$

by Proposition 2.1. The second point is that since the  $R_\alpha$  extend to bounded operators on  $L_p(G)$  one can estimate the right hand norm uniformly for  $\nu, \varepsilon \in \langle 0, 1 \rangle$  by a variation of the argument of [ERS97].

The starting point is the observation that

$$\|R_{\alpha;\nu,\varepsilon}\|_{p \rightarrow p} \leq \|R_\alpha\|_{p \rightarrow p} \|H^{|\alpha|/m}(\nu I + H)^{-|\alpha|/m}(I + \varepsilon H)^{-n}\|_{p \rightarrow p}$$

where for brevity we have set  $\|\cdot\|_{L_p(G) \rightarrow L_p(G)} = \|\cdot\|_{p \rightarrow p}$ . But  $|\alpha|/m = k + \gamma$  with  $k \in \mathbf{N}_+$  and  $\gamma \in [0, 1)$ . Therefore

$$\|R_{\alpha;\nu,\varepsilon}\|_{p \rightarrow p} \leq \|R_\alpha\|_{p \rightarrow p} (\|H(\nu I + H)^{-1}\|_{p \rightarrow p})^k \|H^\gamma(\nu I + H)^{-\gamma}(I + \varepsilon H)^{-n}\|_{p \rightarrow p}.$$

Now consider the case  $\gamma = 0$ . First one has

$$\|H(\nu I + H)^{-1}\|_{p \rightarrow p} \leq \left\| \nu \int_0^\infty dt e^{-\nu t} (I - S_t) \right\|_{p \rightarrow p} \leq b_p$$

with  $b_p = \sup_{t>0} \|I - S_t\|_{p \rightarrow p}$ , which is finite because  $S$  is uniformly bounded by (3) of [ERS97]. Secondly,

$$\|(I + \varepsilon H)^{-n}\|_{p \rightarrow p} \leq \Gamma(n)^{-1} \int_0^\infty dt e^{-t} t^{-1+n} \|S_{\varepsilon t}\|_{p \rightarrow p} \leq a_p$$

where  $a_p = \sup_{t>0} \|S_t\|_{p \rightarrow p}$  which is again finite. Combining these estimates one has

$$\|R_{\alpha;\nu,\varepsilon}\|_{p \rightarrow p} \leq a_p b_p^k \|R_\alpha\|_{p \rightarrow p} \quad (5)$$

uniformly for  $\nu, \varepsilon > 0$ .

Next consider the case  $\gamma \in \langle 0, 1 \rangle$ . Then  $H(\nu I + H)^{-1}$  is bounded and generates a uniformly bounded semigroup. Therefore its fractional power is given by the algorithm

$$H^\gamma(\nu I + H)^{-\gamma} = n_\gamma^{-1} \int_0^\infty d\lambda \lambda^{-1+\gamma} (1 + \lambda)^{-1} H(\lambda(1 + \lambda)^{-1} \nu I + H)^{-1}$$

where  $n_\gamma = \int_0^\infty d\lambda \lambda^{-1+\gamma} (1 + \lambda)^{-1}$ . Hence

$$\|H^\gamma(\nu I + H)^{-\gamma}(I + \varepsilon H)^{-n}\|_{p \rightarrow p} \leq \sup_{\nu, \varepsilon > 0} \|H_\nu(I + H_\nu)^{-1}(I + \varepsilon H)^{-n}\|_{p \rightarrow p}$$

where  $H_\nu = \nu^{-1}H$ . But  $H_\nu$  generates the uniformly bounded semigroup  $S_t^\nu = S_{\nu^{-1}t}$ . Therefore

$$H_\nu(I + H_\nu)^{-1}(I + \varepsilon H)^{-n} = \Gamma(n)^{-1} \int_0^\infty ds \int_0^\infty dt e^{-(s+t)} s^{-1+n} S_{\varepsilon s}(I - S_{\nu^{-1}t}) .$$

Consequently one deduces that

$$\|H_\nu(I + H_\nu)^{-1}(I + \varepsilon H)^{-n}\|_{p \rightarrow p} \leq a_p$$

where  $a_p$  is now given by  $a_p = \sup_{s,t>0} \|S_s - S_t\|_{p \rightarrow p}$ . It then follows from combination of these estimates that (5) is again valid with the new choice of  $a_p$ .

Finally, if  $\varphi \in L'_{p;\infty}(X)$  then

$$\begin{aligned} \|D^\alpha \varphi\|_{L_p(X)} &= \|R_{\alpha;\nu,\varepsilon}^U (I + \varepsilon H_D)^n (\nu I + H_D)^{|\alpha|/m} \varphi\|_{L_p(X)} \\ &\leq \|R_{\alpha;\nu,\varepsilon}\|_{p \rightarrow p} \|(I + \varepsilon H_D)^n (\nu I + H_D)^{|\alpha|/m} \varphi\|_{L_p(X)} \\ &\leq a_p b_p^k \|R_\alpha\|_{p \rightarrow p} \|(I + \varepsilon H_D)^n (\nu I + H_D)^{|\alpha|/m} \varphi\|_{L_p(X)} \end{aligned}$$

where we have used (4) and (5). Therefore taking the limits  $\nu, \varepsilon \rightarrow 0$  one deduces that

$$\|D^\alpha \varphi\|_{L_p(X)} \leq a_p b_p^k \|R_\alpha\|_{p \rightarrow p} \|H_D^{|\alpha|/m} \varphi\|_{L_p(X)} .$$

But these bounds then extend to all  $\varphi \in D(H_D^{|\alpha|/m})$  by continuity (see [ERS97], proof of Lemma 4.2). One immediately deduces that the Riesz transforms extend to bounded operators and that (3) is valid.  $\square$

The theorem has a number of corollaries similar to those given in [ERS97] for the operator  $H$  on the spaces  $L_p(G)$ .

**Corollary 2.4** *For each  $n \in \mathbf{N}$  and  $p \in \langle 1, \infty \rangle$  one has  $D(H_D^{n/m}) = L'_{p;n}(X)$ . Moreover, there are  $c_{p,n}, c'_{p,n} > 0$  such that*

$$c_{p,n} \max_{|\alpha|=n} \|D^\alpha \varphi\|_{L_p(X)} \leq \|H_D^{n/m} \varphi\|_{L_p(X)} \leq c'_{p,n} \max_{|\alpha|=n} \|D^\alpha \varphi\|_{L_p(X)}$$

for all  $\varphi \in L'_{p;n}(X)$ .

**Proof** The left hand bound is just a restatement of the boundedness of the Riesz transforms and the proof of the right hand bound is identical to the proof of the analogous result in Corollary 4.3 of [ERS97].  $\square$

**Corollary 2.5** *The operators  $D^\alpha H_D^{-(|\alpha|+|\beta|)/m} D^\beta$  extend to bounded operators on  $L_p(X)$  for each  $p \in \langle 1, \infty \rangle$ .*

**Proof** This follows by the argument used to prove Corollary 4.6 of [ERS97].  $\square$

The last corollary could also be deduced by transference from the result for  $H$ . The key point is the identity

$$D^\alpha K_U D^\beta = (-1)^{|\beta|} \int_G dg (A^\alpha B^{\beta*} k)(g) U(g) \quad (6)$$

where the  $B_i$  are the generators of right translations of  $G$  and  $\beta_*$  is the multi-index obtained from  $\beta$  by reversing its order. This identity is a generalization of (2) and is proved by a similar argument. Note that the semigroup  $S^U$  acting on the  $L_p(X)$ -spaces can be represented by a distributional kernel  $(x, y) \in X \times X \mapsto K_t(x; y)$  and then (6) gives

$$\int_X dy (D_x^\alpha D_y^\beta K_t)(x; y) \varphi(y) = (-1)^{|\beta|} \int_G dg (A^\alpha B^{\beta*} k)(g) (U(g) \varphi)(x)$$

where  $D_x$  and  $D_y$  indicate that the operators  $D$  act on the  $x$  and  $y$  variables respectively.

Finally one has a weak-type  $(1, 1)$  statement.

**Corollary 2.6** *The Riesz transforms  $R_\alpha^U$  extend to operators of weak-type  $(1, 1)$ .*

This also follows by transference, although it can be deduced in various other ways. Proposition 4.7 of [ERS97] establishes by standard singular integration arguments (see [Ste93], Chapter 1) that the  $R_\alpha$  extend to operators of weak-type  $(1, 1)$ . Then the corollary follows from the second transference theorem of Coifman and Weiss, [CW71] Theorem 2.6. We omit the details.

### 3 Grušin operators

In this section we apply Theorem 2.3 to the Grušin operator defined by (1) in the introduction. But we begin by filling in some of the preliminary details outlined in Section 1.

If  $\nu = (\nu_1, \dots, \nu_n) \in \mathbf{N}_+^n$  is an  $n$ -tuple of positive integers we set  $x^\nu = x_1^{\nu_1} \dots x_n^{\nu_n}$  for each  $x \in \mathbf{R}^n$ . Further let  $I_n(N) = \{\nu : |\nu| = N\}$  where  $|\nu| = \nu_1 + \dots + \nu_n = N$  and set  $\nu! = \nu_1! \dots \nu_n!$ . The binomial expansion gives

$$|x|^{2N} = \sum_{\nu \in I_n(N)} (a_\nu x^\nu)^2$$

with  $a_\nu = (N!/\nu!)^{1/2}$ . Therefore

$$H_N = - \sum_{j=1}^n X_j^2 - \sum_{k=1}^m \sum_{\nu \in I_n(N)} X_{\nu,k}^2$$

with  $X_j = \partial_{x_j}$  and  $X_{\nu,k} = a_\nu (x^\nu \partial_{y_k})$ . Thus  $H_N$  is the ‘sum of squares’ of a finite family of first-order operators  $\mathcal{X}_n \cup \mathcal{X}_{N,m}$  where

$$\mathcal{X}_n = \{X_j : 1 \leq j \leq n\} \quad \text{and} \quad \mathcal{X}_{N,m} = \{X_{\nu,k} : \nu \in I_n(N), 1 \leq k \leq m\}.$$

Both sets are abelian but  $[X_j, Y_{\nu,k}] = a_\nu (\partial_{x_j} x^\nu) \partial_{y_k} \in \mathcal{Y}_{N-1,m}$ . Therefore  $\mathcal{X}_n \cup \mathcal{X}_{N,m}$  generates a nilpotent Lie algebra  $\mathfrak{g}_N$  of rank  $N + 1$ . (The generating set has dimension  $d' = n + m \binom{N+n-1}{n-1}$  and the dimension of the Lie algebra is given by  $d = n + m \binom{N+n}{n}$ .) It is now notationally convenient to fix an enumeration of the set  $\mathcal{X}_{N,m}$  and relabel it as  $X_j$  with  $n + 1 \leq j \leq d'$ . Then

$$H_N = - \sum_{j=1}^{d'} X_j^2$$

and the set  $X_1, \dots, X_{d'}$  is an algebraic basis of the nilpotent Lie algebra  $\mathfrak{g}_N$ .

Let  $G_N$  denote the connected simply connected nilpotent Lie group which has  $\mathfrak{g}_N$  as Lie algebra. Since the closure of each  $X_j$  generates a one-parameter group of translations on the spaces  $L_p(\mathbf{R}^{n+m})$  the family of  $X_j$  generate an isometric representation of  $G$  on  $L_p(\mathbf{R}^{n+m})$  for each  $p \in [1, \infty]$ . The Grušin operator  $H_N$  is the corresponding sublaplacian. In particular  $H_N$  is a second-order subcoercive operator (see Remark 2.2). Next if  $s \in \mathbf{N}_+$  and  $\alpha = (i_1, \dots, i_s) \in \{1, \dots, d'\}^s$  we set  $X^\alpha = X_{i_1} \dots X_{i_s}$  and  $|\alpha| = s$ . Then one immediately deduces the following.

**Theorem 3.1** *The Riesz transforms  $X^\alpha H_N^{-|\alpha|/2}$  extend to bounded operators on each of the spaces  $L_p(\mathbf{R}^{n+m})$  with  $p \in \langle 1, \infty \rangle$  and are of weak-type  $(1, 1)$ .*

This is a direct corollary of Theorem 2.3, Corollary 2.6 and the foregoing observations.

Next consider the Riesz transforms associated with the Grušin operator  $H_N$  and the  $n + m$  operators  $Y_j$  given by  $Y_j = \partial_{x_j}$  if  $j \in \{1, \dots, n\}$  and  $Y_j = |x|^N \partial_{y_l}$  if  $j = n + l$  with  $l \in \{1, \dots, m\}$ . Now

$$H_N = - \sum_{j=1}^{n+m} Y_j^2$$

but the  $Y_j$  no longer generate a finite-dimensional Lie algebra. For example,  $[Y_j, Y_{n+l}] = (x_j/|x|) N |x|^{N-1} \partial_{y_l}$ . Nevertheless one has the following partial conclusion about the corresponding Riesz transforms. (In the sequel the multi-index  $\alpha$  corresponds to indices in the set  $\{1, \dots, n + m\}$ .)

**Theorem 3.2** *If  $n \geq 2$  the Riesz transforms  $Y^\alpha H_N^{-|\alpha|/2}$  with  $|\alpha| \leq N + 1$  extend to bounded operators on  $L_p(\mathbf{R}^{n+m})$  for each  $p \in \langle 1, \infty \rangle$  and are of weak-type  $(1, 1)$ .*

**Proof** First consider the case  $|\alpha| = 1$ . If  $j \leq n$  then  $Y_j = X_j$ . But if  $j > n$  then

$$\|Y_j \varphi\|_p = \| |x|^N \partial_{y_j} \varphi \|_p \leq \| |x|_1^N \partial_{y_j} \varphi \|_p$$

where  $|x|_1$  is the  $l_1$ -norm of  $x$ . Then, however, one has an estimate

$$\|Y_j \varphi\|_p \leq c_N \sup_{\nu \in I_n(N)} \|x^\nu \partial_{y_j} \varphi\|_p \leq c_N \sup_{k > n} \|X_k \varphi\|_p.$$

It then follows from Theorem 3.1 and Corollary 2.4 that  $D(H_N^{1/2}) = \bigcap_{j=1}^{n+m} D(Y_j)$  and  $\|Y_j \varphi\|_p \leq c_N \sup_{1 \leq k \leq d'} \|X_k \varphi\|_p \leq c'_N \|H_N^{1/2} \varphi\|_p$  for all  $j \in \{1, \dots, n + m\}$ ,  $\varphi \in D(H_N^{1/2})$  and  $p \in \langle 1, \infty \rangle$ . Thus the first-order transforms  $Y_j H_N^{-1/2}$  extend to bounded operators on the  $L_p$ -spaces and are weak-type  $(1, 1)$  by Theorem 3.1 and Corollary 2.6.

Next we proceed by induction. We make the induction hypothesis that

$$\sup_{\{\alpha: |\alpha|=M\}} \|Y^\alpha \varphi\|_p \leq c_N \sup_{\{\beta: |\beta|=M\}} \|X^\beta \varphi\|_p \quad (7)$$

for all  $\varphi \in C_c^\infty(\mathbf{R}^{n+m})$  and some  $M \leq N$ . It follows from the foregoing that the hypothesis is valid for  $M = 1$  and we use this to argue that it is also valid for  $M + 1$ .

Let  $Y^\alpha = Y^\gamma Y_j$  with  $|\alpha| = M + 1$ . If  $j < n$  then  $Y_j = X_j$  and

$$\|Y^\alpha \varphi\|_p = \|Y^\gamma Y_j \varphi\|_p \leq c_N \sup_{\{\gamma: |\gamma|=M\}} \|X^\gamma X_j \varphi\|_p \leq c_N \sup_{\{\beta: |\beta|=M+1\}} \|X^\beta \varphi\|_p$$



and (7) is satisfied. Therefore assume that  $n < j \leq n + m$ . Then

$$\begin{aligned} \|Y^\alpha \varphi\|_p &\leq c_N \sup_{\{\gamma: |\gamma|=M\}} \|X^\gamma Y_j \varphi\|_p \\ &\leq c_N \sup_{\{\gamma: |\gamma|=M\}} \|Y_j X^\gamma \varphi\|_p + c_N \sup_{\{\gamma: |\gamma|=M\}} \|(\text{ad } X^\gamma)(Y_j) \varphi\|_p. \end{aligned} \quad (8)$$

The first-order estimate then gives an upper bound  $c'_N \sup_{\{\beta: |\beta|=M+1\}} \|X^\beta \varphi\|_p$  on the first term on the right as required. It remains to estimate the second term.

The starting point is the multi-commutator algorithm

$$(\text{ad } X^\gamma)(Y_j) = \sum_{\delta \subseteq \gamma} (\text{ad } X)^\delta(Y_j) X^{\gamma \setminus \delta}. \quad (9)$$

The sum is over all non-empty ordered subsets  $\delta$  of  $\gamma$  and  $\gamma \setminus \delta$  is the ordered set obtained from  $\gamma$  by omitting the elements of  $\delta$ . Now by assumption  $Y_j = |x|^N \partial_{y_j}$  and each  $X_j$  is either of the form  $x^\nu \partial_{y_k}$  or  $\partial_{x_l}$ . But each  $x^\nu \partial_{y_k}$  commutes with  $Y_j$  and with each  $(\text{ad } \partial_x)^{\delta'}(Y_j)$ . The latter observation follows because  $(\text{ad } \partial_x)^{\delta'}(Y_j) = (\partial_x^{\delta'} |x|^N) \partial_{y_j}$ . Therefore the only non-zero terms on the right hand side of (9) correspond to multi-indices  $\delta' \subseteq \gamma$  with entries from the set  $\{1, \dots, n\}$ . Therefore

$$(\text{ad } X^\gamma)(Y_j) = \sum_{\delta' \subseteq \gamma} (\text{ad } X)^{\delta'}(Y_j) X^{\gamma \setminus \delta'} = \sum_{\delta' \subseteq \gamma} (\partial_x^{\delta'} |x|^N) \partial_{y_j} X^{\gamma \setminus \delta'}.$$

Now since  $|\delta'| \leq |\gamma| \leq N$  and  $n \geq 2$  there are  $a_N, b_N > 0$  such that

$$|(\partial_x^{\delta'} |x|^N)| \leq a_N |x|^{N-|\delta'|} \leq b_N \sup_{\{\nu: |\nu|=N-|\delta'|\}} |x^\nu| \leq b_N \sup_{\{\nu: |\nu|=N\}} |(\partial_x^{\delta'} x^\nu)|.$$

Consequently,

$$|(\text{ad } X)^{\delta'}(Y_j) X^{\gamma \setminus \delta'} \varphi| = |(\partial_x^{\delta'} |x|^N) \partial_{y_j} X^{\gamma \setminus \delta'} \varphi| \leq b_N \sup_{\{\nu: |\nu|=N\}} |(\partial_x^{\delta'} x^\nu) \partial_{y_j} X^{\gamma \setminus \delta'} \varphi|$$

But  $x^\nu \partial_{y_j} = X_l$  for a suitable choice of  $l \in \{1, \dots, n + m\}$  and since  $\delta'$  has entries in  $\{1, \dots, n\}$  one has

$$(\partial_x^{\delta'} x^\nu) \partial_{y_j} X^{\gamma \setminus \delta'} = (\text{ad } X)^{\delta'}(X_l) X^{\gamma \setminus \delta'}.$$

As the right hand expression is a sum of monomials of order  $|\gamma| + 1$  in the  $X_j$  one concludes that

$$\|(\text{ad } X)^{\delta'}(Y_j) X^{\gamma \setminus \delta'} \varphi\|_p \leq c_N \sup_{\{\beta: |\beta|=|\gamma|+1\}} \|X^\beta \varphi\|_p.$$

Finally it follows from (9) that

$$\sup_{\{\gamma: |\gamma|=M\}} \|(\text{ad } X^\gamma)(Y_j) \varphi\|_p \leq c'_N \sup_{\{\beta: |\beta|=M+1\}} \|X^\beta \varphi\|_p$$

and this gives the required bound on the second term on the right of (8). This completes the induction.

One concludes that (7) is valid for all  $M \in \{1, \dots, N + 1\}$  and all  $p \in \langle 1, \infty \rangle$ . Then the boundedness of the  $Y$ -transforms of all orders up to  $N + 1$  follows from the boundedness of the  $X$ -transforms. The weak-type  $(1, 1)$  property follows similarly with the aid of Corollary 2.6.  $\square$

The  $Y$ -transforms of order larger than  $N + 1$  are not necessarily bounded. Nevertheless the foregoing argument establishes that a large subset of the transforms are bounded.

**Corollary 3.3** *Assume  $n \geq 2$ . Let  $Y^\alpha = Y^\gamma Y_l$  with  $|\alpha| = N + 2$ . If either  $l \leq n$  or  $l > n$  and  $\max\{\gamma_j : \gamma_j \in \gamma\} > n$  then  $Y^\alpha H_N^{-|\alpha|/2}$  extends to a bounded operator on  $L_p(\mathbf{R}^{n+m})$  for each  $p \in \langle 1, \infty \rangle$  and is of weak-type  $(1, 1)$ .*

**Proof** If  $l \leq n$  then  $Y_l = X_l$ . Hence  $\|Y^\alpha \varphi\|_p \leq \sup_{\{\gamma: |\gamma|=N+1\}} \|X^\gamma X_l \varphi\|_p$  and the conclusion follows as before.

If  $l > n$  and  $\max\{\gamma_j : \gamma_j \in \gamma\} > n$  then one uses (8) and argues as above. The assumption on the indices of  $\gamma$  ensures that there are at most  $N$  derivatives  $\partial_x$  of  $|x|^N$  occurring in the estimates. Since  $n \geq 2$  all terms can be bounded as before.  $\square$

The  $Y$ -transforms of order  $N + 2$  are, however, bounded for small values of  $p$ .

**Proposition 3.4** *If  $n \geq 2$  the Riesz transforms  $Y^\alpha H^{-|\alpha|/2}$  with  $|\alpha| = N + 2$  extend to bounded operators on  $L_p(\mathbf{R}^{n+m})$  for all  $p \in \langle 1, n \rangle$  and are weak-type  $(1, 1)$ .*

*Conversely, if  $n \geq 1$  then the transforms corresponding to  $Y^\alpha = Y^\gamma Y_l$  with  $\gamma \in \{1, \dots, n\}^{N+1}$  and  $l > n$  do not extend to bounded operators on  $L_p(\mathbf{R}^{n+m})$  for  $p \in [n, \infty)$ .*

**Proof** Consider the first statement. As a consequence of Corollary 3.3 it suffices to prove boundedness for  $Y^\alpha = Y^\gamma Y_l$  with  $\gamma \in \{1, \dots, n\}^{N+1}$  and  $l > n$ . But then  $Y^\alpha = X^\gamma Y_l$  and

$$Y^\alpha \varphi = Y_l X^\gamma \varphi + \sum'_{\delta \subseteq \gamma} (\text{ad } X)^\delta (Y_l) X^{\gamma \setminus \delta} \varphi$$

where the prime indicates that the sum is restricted to multi-indices  $\delta$  with entries in  $\{1, \dots, n\}$ . Now arguing as in the proof of Theorem 3.2 the  $L_p$ -norms of all terms on the right can be bounded by a multiple of  $\sup_{\{\beta: |\beta|=N+2\}} \|X^\beta \varphi\|_p$  with the exception of the leading term in the commutator sum, the term  $(\text{ad } X)^\gamma (Y_l) \varphi$ . These latter estimates are valid for all  $p \in \langle 1, \infty \rangle$ . Now consider the  $L_p$ -norm of the exceptional term.

Since  $|\gamma| = N + 1$  and  $n \geq 2$  one has

$$\|(\text{ad } X)^\gamma (Y_l) \varphi\|_p = \|\partial_x^\gamma (|x|^N) \partial_{y_l} \varphi\|_p \leq c_N \| |x|^{-1} \partial_{y_l} \varphi \|_p. \quad (10)$$

Next we use the multi-dimensional  $L_p$ -version of the classical Hardy inequality (see, for example, [OK90] Chapter 2 or [Maz11], Section 1.3.1). Explicitly, if  $1 \leq p < n$  then

$$\int_{\mathbf{R}^n} dx |x|^{-p} |\varphi(x)|^p \leq (p/(n-p))^p \int_{\mathbf{R}^n} dx |(\nabla_x \varphi)(x)|^p \quad (11)$$

for all  $\varphi \in C_c^\infty(\mathbf{R}^n)$ . Combining (10) and (11) one finds there is a  $c_{p,N} > 0$  such that

$$\|(\text{ad } X)^\gamma (Y_l) \varphi\|_p \leq c_{p,N} \sup_{1 \leq j \leq n} \|\partial_{x_j} \partial_{y_l} \varphi\|_p.$$

But  $\partial_{y_l} = (\text{ad } X^\delta)(X_k)$  for a suitable choice  $\delta \in \{1, \dots, n\}^N$  and  $k > n$ . In addition  $\partial_{x_j} = X_j$ . Therefore one has an estimate

$$\|(\text{ad } X)^\gamma (Y_l) \varphi\|_p \leq a_{p,N} \sup_{\{\beta: |\beta|=N+2\}} \|X^\beta \varphi\|_p.$$

Hence

$$\|Y^\alpha \varphi\|_p \leq b_{p,N} \sup_{\{\beta: |\beta|=N+2\}} \|X^\beta \varphi\|_p$$

with  $b_{p,N} > 0$ . It then follows from the boundedness of the  $X$ -transforms on  $L_p(\mathbf{R}^{n+m})$  with  $p \in \langle 1, \infty \rangle$  that the  $Y^\alpha H^{-|\alpha|/2}$  extend to bounded operators on the  $L_p$ -spaces with  $p \in \langle 1, n \rangle$ . The weak-type  $(1, 1)$  property can be established similarly.

Next consider the converse statement. Since  $\gamma \in \{1, \dots, n\}^{N+1}$  and  $Y_j = X_j$  for  $j \leq n$  it follows that  $Y^\alpha = X^\gamma Y_l$ . But a necessary condition for the transforms to extend to bounded operators is the inclusion  $C_c^\infty(\mathbf{R}^{n+m}) \subseteq D(Y^\alpha)$ . Therefore we argue that there is a  $\varphi \in C_c^\infty(\mathbf{R}^{n+m})$  such that  $\varphi \notin D(Y^\alpha)$ . Let  $\varphi = \varphi_1 \varphi_2$  and  $\psi = \psi_1 \psi_2$  with  $\varphi_1, \psi_1 \in C_c^\infty(\mathbf{R}^n)$  and  $\varphi_2, \psi_2 \in C_c^\infty(\mathbf{R}^m)$ . Then

$$(-1)^{N+1}(X^\gamma \psi, Y_l \varphi) = (-1)^{N+1}(\partial_x^\gamma \psi_1, |x|^N \varphi_1)(\psi_2, \partial_{y_l} \varphi_2).$$

Hence for the inclusion  $\varphi \in D(Y^\alpha)$  on  $L_p(\mathbf{R}^{n+m})$  it is necessary that  $\psi \mapsto (\partial_x^\gamma \psi_1, |x|^N \varphi_1)$  is  $L_q$ -continuous where  $q$  is the conjugate to  $p$ . But  $|\gamma| = N + 1$  and if  $\varphi > 0$  in an open neighbourhood of the origin  $L_q$ -continuity requires that  $p < n$ .  $\square$

One may similarly deduce boundedness of higher order  $Y$ -transforms for a suitable range of  $p$  and  $n$ . But then the Hardy inequality has to be replaced by the Rellich inequality or an appropriate higher order generalization. For example, if  $|\alpha| = N + 3$  then the critical transform to bound is  $Y^\alpha = Y^\gamma Y_l$  with  $\gamma \in \{1, \dots, n\}^{N+2}$  and  $l > n$ . Then (10) is replaced by an estimate

$$\|(\text{ad } X)^\gamma(Y_l)\varphi\|_p \leq c_N \| |x|^{-2} \partial_{y_l} \varphi \|_p. \quad (12)$$

But if  $n \geq 3$  and  $p \in \langle 1, n/2 \rangle$  then the Rellich inequality (see, for example, (3) in [DH98]) states that

$$\int_{\mathbf{R}^n} dx |x|^{-2p} |\varphi(x)|^p \leq c_{p,n}^p \int_{\mathbf{R}^n} dx |(\Delta_x \varphi)(x)|^p \quad (13)$$

for all  $\varphi \in C_c^\infty(\mathbf{R}^n)$  with  $c_{p,n} = p^2 / ((p-1)n(n-2p))$ . Arguing as above but with (11) replaced by (13) one concludes the following.

**Corollary 3.5** *If  $n \geq 3$  the Riesz transforms  $Y^\alpha H^{-|\alpha|/2}$  with  $|\alpha| = N+3$  extend to bounded operators on  $L_p(\mathbf{R}^{n+m})$  for all  $p \in \langle 1, n/2 \rangle$  and are weak-type  $(1, 1)$ .*

Again the upper bound  $n/2$  is optimal. If  $p \geq n/2$  there are some transforms of order  $N + 3$  which are unbounded.

The foregoing methods extend to operators  $-\nabla_x^2 - c(x) \nabla_y^2$  with  $c$  a sum of positive multiples of even powers of  $|x|$ . But they do not shed light on more general situations such as  $c(x) = |x|^\delta$  with  $\delta > 0$  but  $\delta \neq 2N$ . In particular it would be of interest to understand the properties of Riesz transforms in the case  $c(x) = |x|$  analyzed in [CS12]. It is also not clear if the  $|x|^{2N}$ -results for are still valid if one only has  $c(x) \sim |x|^{2N}$ , i.e. if  $a|x|^{2N} \leq c(x) \leq b|x|^{2N}$  for some  $a, b > 0$ , although many properties of Grushin operators are known to be invariant under such equivalence relations [RS08] [RS14].

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